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DIRECT DERIVATION OF THE ORDINARY CANONICAL SYSTEM OF ELLIPTIC ELEMENTS EMPLOYED IN THE PROBLEM OF THREE BODIES.

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The canonical equations of motion may be written

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad (1)$$

in which

$$H = \Sigma p_i \frac{dq_i}{dt} - T - F \quad (2)$$

is a constant, and

$$p_i = \frac{\partial T}{\partial \frac{dq_i}{dt}}. \quad (3)$$

T is a function of the q_i 's and $\frac{dq_i}{dt}$'s but H is expressed as a function of the p_i 's and q_i 's.

We will consider the problem of three bodies in which

$$T \equiv \frac{1}{2} \left[\frac{dr}{dt} \right]^2 + \frac{1}{2} r^2 \left[\frac{dw}{dt} \right]^2, \quad (4)$$

$$F \equiv \frac{\mu}{r} + R_1, \quad (5)$$

$$H = \frac{1}{2} \left[\frac{dr}{dt} \right]^2 + \frac{1}{2} r^2 \left[\frac{dw}{dt} \right]^2 - \frac{\mu}{r} - R_1, \quad (6)$$

where r , w are the radius vector and longitude in orbit of the disturbed body, R_1 is a function of the coordinates of both the disturbed and disturbing bodies, and μ is a constant.

As q_i 's we will select

$q_1 \equiv$ the mean anomaly,

$q_2 \equiv$ the angular distance of the perihelion from the node,

$q_3 \equiv$ the longitude of the node counted from a fixed point in the fundamental plane; in each case reference being had to the instantaneous Keplerian ellipse. It remains to find p_1, p_2, p_3 .

dq_1/dt is independent of dq_2/dt and dq_3/dt ; it is also independent of the form of the orbit. In deriving p_1 , therefore, by means of (3), we may write T_0 in the place of T , the former being obtained on the assumption that the orbit

is a circle whose radius is the semi-major axis, a , of the instantaneous ellipse. In the instantaneous ellipse, instead of (6) we have

$$\frac{1}{2} \left[\frac{dr}{dt} \right]^2 + \frac{1}{2} r^2 \left[\frac{dw}{dt} \right]^2 - \frac{\mu}{r} + \frac{\mu}{2a} = 0; \quad (7)$$

In the corresponding circle $dr/dt = 0$, $r = a$, $dw/dt = dq_1/dt$; whence

$$T_0 = \frac{1}{2} a^2 \left[\frac{dq_1}{dt} \right]^2, \quad (8)$$

and

$$p_1 = \frac{\partial T_0}{\partial \frac{dq_1}{dt}} = a^2 \frac{dq_1}{dt}. \quad (9)$$

Also, from (7), on the same assumption,

$$\frac{1}{2} a^2 \left[\frac{dq_1}{dt} \right]^2 - \frac{\mu}{2a} = 0, \quad (10)$$

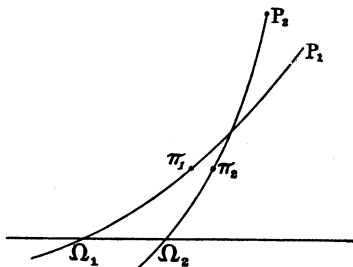
or

$$\frac{dq_1}{dt} = \sqrt{\frac{\mu}{a^3}}. \quad (11)$$

Combining (9) and (11), we have

$$p_1 = \sqrt{\mu a}. \quad (a)$$

A reference to the accompanying figure (in which P_1, P_2 are successive positions of the disturbed body on the celestial sphere, and Ω_1, Ω_2 ; π_1, π_2 are



the corresponding positions of the instantaneous node and perihelion) will show that

$$\frac{dw}{dt} = \frac{dv}{dt} + \frac{dq_2}{dt} + \frac{dq_3}{dt} \cos i, \quad (12)$$

where v is the instantaneous true anomaly, and i is the inclination of the

instantaneous orbit to the fundamental plane; and since dq_2/dt is independent of dr/dt , dv/dt , and dq_3/dt , equations (3), (4), and (12) give

$$p_2 = \frac{\partial T}{\partial \frac{dq_2}{dt}} = r^2 \frac{dw}{dt}. \quad (13)$$

Substituting in (7),

$$\frac{1}{2} \left[\frac{dr}{dt} \right]^2 + \frac{1}{2} \frac{p_2^2}{r^2} - \frac{\mu}{r} + \frac{\mu}{2a} = 0. \quad (14)$$

This equation holds true for all points of the instantaneous ellipse. For the maximum and minimum values of r , since for these values $dr/dt = 0$, we have

$$\frac{1}{2} \frac{p_2^2}{r^2} - \frac{\mu}{r} + \frac{\mu}{2a} = 0,$$

or

$$r^2 - 2ar + \frac{a}{\mu} p_2^2 = 0. \quad (15)$$

Calling the roots of this equation $r_1 = a(1 - e)$ and $r_2 = a(1 + e)$, as given by the properties of the ellipse, the last term of (15) gives

$$r_1 r_2 = \frac{a}{\mu} p_2^2 = a^2(1 - e^2),$$

or

$$p_2 = \sqrt{\mu a(1 - e^2)} = p_1 \sqrt{1 - e^2}, \quad (b)$$

in which, of course, e is the eccentricity of the instantaneous ellipse.

Finally, since dq_3/dt is independent of dr/dt , dv/dt , dq_2/dt , equations (3), (4), and (12) give

$$p_3 = \frac{\partial T}{\partial \frac{dq_3}{dt}} = r^2 \frac{dw}{dt} \cos i = p_2 \cos i. \quad (c)$$

A comparison of (7) with (6) gives

$$H = -\frac{\mu}{2a} - R_1 = -\frac{\mu^2}{2p_1^2} - R_1 = -R,$$

in which R is the expression used by Delaunay in his *Théorie de la Lune*.

The derivation of the ordinary canonical system here given has the following apparent advantages: 1. The use of Hamilton's principal function is avoided; 2. The argument forming Chap. 36 of Jacobi's *Vorlesungen über Dynamik*, or that forming Vol. I, Art. 59, of Tisserand's *Mécanique Céleste*, is rendered unnecessary in applying the system to the theory of perturbations; 3. The reason for the addition of the term $\mu^2/(2p_1^2)$ to the perturbative function is shown, without a special investigation, such as that given in Vol. I, Art. 5, of Delaunay's *Théorie de la Lune*.